Harmonic solutions and weak solutions of two-dimensional incompressible Euler equations with Coriolis force

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This article is dedicated to Prof. Richard Askey.

Abstract

In this paper, two families of exact solutions to two-dimensional incompressible Euler equations with Coriolis force are constructed by connecting the classical Euler equations with the Laplace equation via a stream function. The constituent solutions in the first family are smooth, orthogonal, and conjugate harmonic solutions, while their constituent velocities are nonlinear with respect to the spatial variables. The second family are weak solutions in the distribution sense.

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1 Introduction

The Navier-Stokes equations are in vector form. In the general case,

$$\vec{u}_t + (\vec{u} \cdot \triangledown)\vec{u} + \triangledown p = \mu \Delta \vec{u}, \tag{1.1}$$

where $\vec{u}$ is the fluid velocity, t is time, $\triangledown$ is the Hamilton operator, p is the ratio of the pressure to the the fluid density, μ is the kinematic viscosity, and Δ is the Laplace operator. Being the fundamental model for fluid motions in hydromechanics and electromagnetic fields, this system has been extensively studied. Various special and simpler models of system (1.1) have also been studied [1] - [14].

Under the additional condition $\mathrm{div}(\vec{u}) = 0$, the flow is said to be incompressible. Furthermore, when the viscosity $\mu = 0$ (or is small enough to be neglected), the Navier-Stokes equations are reduced to the Euler equations [9–15]

$$\mathrm{div}(\vec{u}) = 0, \tag{1.2a}$$

$$\vec{u}_t + (\vec{u} \cdot \triangledown)\vec{u} + \triangledown p = 0. \tag{1.2b}$$

Constructing exact solutions for physical models is a common and natural activity. In 1965, Arnold [4] first introduced the famous Arnold-Beltrami-Childress (ABC) flow for the stationary three-dimensional (3D) Euler equations. Although the solutions have infinite total energy, they provide local information that is useful for understanding the great complexity in turbulence (see, for example, Ludlow et al. [5]). Zelik [6, 7] studied the existence of weak solutions for the unbounded domain. In 1995, Zhang and Zheng [16] obtained spiral solutions for the 2D compressible Euler equations under the condition $p = K\rho^{\gamma}$, where $\gamma = 2$. In 2011, Yuen [17] obtained a class of exact and rotational solutions for the 3D compressible and incompressible Euler equations. In 2012-2015, Yuen [18] - [20] constructed a family of exact solutions for the compressible Euler equations by using a new characteristic method. In 2015, An, Fan and Yuen [21] applied the Cartesian vector solution method to obtain general results for the linear flows of the compressible Euler equations. In many of these recent studies, the solution $\vec{u}$ has linear spatial variables x, y, and z. A further extension of this reaearch is to look for novel nonlinear solutions. Later, Fan, Yuen, and An et.al [15, 22] found such solutions for the 2D incompressible Euler equations without and with Coriolis force using the Clarkson-Kruskal (CK) reduction method. We further developed this method to the 3D incompressible Euler equations [23].

In this paper, we consider 2D incompressible Euler equations [22] with Coriolis force, with the solution written in the component notation $\vec{u} = (u, v)$.

$$u_x + v_y = 0, \tag{1.3a}$$

$$u_t + uu_x + vu_y + p_x - kv = 0, \tag{1.3b}$$

$$v_t + uv_x + vv_y + p_y + ku = 0, \tag{1.3c}$$

where $k \neq 0$ is the Coriolis parameter. System (1.3) can be used to model large-scale geophysical motions in a thin layer of fluid under the influence of the Coriolis rotational force [24, 25]. When the Coriolis parameter $k = 0$, system (1.3) is reduced to the classical Euler equations.

Our goal is to find exact solutions of a special form. Specifically, we assume that there exists a "stream function" ϕ such that [3]

$$u = -\phi_y, \quad v = \phi_x, \tag{1.4}$$

and

$$\phi_{xx} + \phi_{yy} = 0. \tag{1.5}$$

In Section 2, we justify the existence of such solutions. By exploiting this fact, in Sections 3 and 4, two families of physically interesting exact solutions for the Euler equations with Coriolis force (1.3) are constructed. These solutions are nonlinear with respect to x and y. The solution of the first family is smooth, and the components are a pair of orthogonal conjugate harmonic functions. The solution of the second family is weak in the distribution sense.

2 Properties of 2D solutions

Theorem 1 *Assuming the existence of a solution of the form (1.4), define*

$$\Omega = \phi_{xx} + \phi_{yy}. \tag{2.6}$$

Then

$$\Omega_t + (\phi_x \Omega_y - \phi_y \Omega_x) = 0. \tag{2.7}$$

Proof.

By our assumption (1.4), (1.3a) is automatically satisfied, and (1.3b)–(1.3c) are reduced to

$$\phi_{yt} - \phi_y\phi_{xy} + \phi_x\phi_{yy} - p_x + k\phi_x = 0, \tag{2.8a}$$

$$\phi_{xt} - \phi_y\phi_{xx} + \phi_x\phi_{xy} + p_y - k\phi_y = 0. \tag{2.8b}$$

Applying the compatible condition $p_{xy} = p_{yx}$ to these two equations yields

$$(-\phi_{yt} + \phi_y\phi_{xy} - \phi_x\phi_{yy})_y = (\phi_{xt} - \phi_y\phi_{xx} + \phi_x\phi_{xy})_x, \tag{2.9}$$

which in turn yields

$$(\phi_{xx} + \phi_{yy})_t - \phi_y(\phi_{xx} + \phi_{yy})_x + \phi_x(\phi_{xx} + \phi_{yy})_y = 0, \tag{2.10}$$

which is equivalent to (2.7). ■

An obvious choice of Ω that satisfies (2.7) is the trivial solution $\Omega \equiv 0$, and (2.6) becomes (1.5). Hence, an easy corollary of Theorem 1 is as follows.

Corollary 1 *For any solution ϕ of (1.5), (u, v) given by (1.4) is a solution of the 2D incompressible Euler equations with Coriolis force (1.3) for some suitable pressure function p.*

Theorem 2 *The solution (u, v) that is constructed above satisfies the classic Cauchy-Riemann equation, i.e., it is a pair of conjugate harmonic functions. As a result, the following statements are true.*

(i) *$f(z) = v(x, y) + iu(x, y)$ is an analytic function of $z = x + iy$.*

(ii) *For any pair of constants c_1 and c_2, the level/contour curves (so-called streamlines of the flow) $u(x, y) = C_1$ and $v(x, y) = C_2$ are orthogonal to each other. In other words, each $u(x, y) = C_1$ is the curve of the steepest descent/ascent of $v(x, y) = C_2$.*

(iii) *The 2D incompressible Euler system with Coriolis force has no non-constant bounded conjugate harmonic solutions.*

Proof. These are all elementary facts in the theory of analytic functions of a complex variable. ■

3 Nonlinear smooth solutions

The Laplace equation (2.6) admits two solutions in exponential form

$$\phi = \exp(\alpha y \pm i\alpha x), \tag{3.1}$$

where $\alpha = \alpha(t)$ is an arbitrary real function of t. By applying the principle of superposition, we obtain the general solution in the form

$$\phi = e^{\alpha y}[c_1 \cos(\alpha x) + c_2 \sin(\alpha x)], \tag{3.2}$$

where $c_1 = c_1(t)$ and $c_2 = c_2(t)$ are arbitrary functions of t.

Substituting (3.2) into (1.4) gives nonlinear smooth solutions to the Euler equation with Coriolis force

$$u = -\alpha e^{\alpha y}[c_1 \cos(\alpha x) + c_2 \sin(\alpha x)], \tag{3.3a}$$

$$v = -\alpha e^{\alpha y}[c_1 \sin(\alpha x) - c_2 \cos(\alpha x)]. \tag{3.3b}$$

It follows from (2.8a) and (2.8b) that p is a complete differential. Therefore, the second kind of curvilinear integral of $p(x, y, t)$ is independent of its integration route. Thus, we can take a special integration path to directly obtain

$$\begin{aligned} p &= \int_0^x p_x(x, 0)dx + \int_0^y p_y(x, y)dy, \\ &= e^{\alpha y}[(c_1\alpha' x - c_2\alpha' y + kc_1 - c_2') \cos(\alpha x) + (c_1\alpha' y + c_2\alpha' x + kc_2 + c_1') \sin(\alpha x)] + \\ &\quad c_2' - kc_1 + \frac{1}{2}(c_1^2 + c_2^2)\alpha^2[1 - e^{2\alpha y}]. \end{aligned} \tag{3.4}$$

Example 1 Choosing $\alpha = -1$, and $c_1 = c_2 = \frac{1}{t}$ in (3.3)-(3.4) gives

$$\begin{aligned} u &= \frac{1}{t}e^{-y}(\cos x - \sin x), \\ v &= -\frac{1}{t}e^{-y}(\sin x + \cos x), \\ p &= \frac{1}{t^2}e^{-y}(\sin x + \cos x + kt\cos x - kt\sin x) - \frac{kt + e^{-2y}}{t^2}. \end{aligned} \tag{3.5}$$

This is a smooth solution that is nonlinear with respect to x and y. Note that the variables are separated. Moreover, there exists a Hilbert transformation between u and v, namely,

$$\begin{aligned} H(u)(x) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(\tau)}{x - \tau} d\tau \\ &= \frac{1}{t}e^{-y}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos\tau + \sin\tau}{x - \tau} d\tau \\ &= -\frac{1}{t}e^{-y}(\sin x + \cos x) \\ &= v(t, x, y). \end{aligned} \tag{3.6}$$

Remark 1 By symmetry, (2.6) also admits solutions in the form of $\phi = e^{\alpha x \pm i\alpha y}$, implying that the Euler equations with Coriolis force also admit solutions in the form of

$$u = \alpha e^{\alpha x}[c_1 \sin(\alpha y) - c_2 \cos(\alpha y)], \tag{3.7a}$$

$$v = \alpha e^{\alpha x}[c_1 \cos(\alpha y) + c_2 \sin(\alpha y)], \tag{3.7b}$$

$$\begin{aligned} p = &-e^{\alpha x}[(c_1 \alpha' y - c_2 \alpha' x - kc_1 - c_2') \cos(\alpha y) + (c_1 \alpha' x + c_2 \alpha' y - kc_2 + c_1') \sin(\alpha y)] - \\ &c_2' - kc_1 + \frac{1}{2}(c_1^2 + c_2^2)\alpha^2[1 - e^{2\alpha x}]. \end{aligned} \tag{3.7c}$$

4 Weak solutions

In this section, we present one kind of singular weak solutions.

Theorem 3 *The Laplace equation (1.5) admits the fundamental solution*

$$\phi = -\frac{1}{2\pi} \ln r - \dot{x}_0 y + \dot{y}_0 x, \ r = \sqrt{(x - x_0)^2 + (y - y_0)^2}, \tag{4.1}$$

that is,

$$-\Delta\phi = \delta(\mathbf{x} - \mathbf{x}_0), \qquad \mathbf{x} = (x, y), \quad \mathbf{x}_0 = (x_0, y_0), \tag{4.2}$$

where $x_0 \equiv x_0(t)$, $y_0 \equiv y_0(t)$ are arbitrary functions of t.

Proof. It is well-known that $\ln r$ is harmonic in $\mathbf{R} \setminus \{O\}$, i.e.,

$$\Delta \ln r = 0, \quad r \neq 0. \tag{4.3}$$

For the case of generalized function, we have

$$\langle -\Delta\phi, \varphi \rangle = \frac{1}{2\pi} \langle \Delta \ln r, \varphi \rangle = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int\int_{r \geq \varepsilon} \ln r \Delta\varphi dx dy. \tag{4.4}$$

Choose a spherical shell $\varepsilon \leq r \leq A$, where A is so large that for $r \geq A$, the function $\varphi(\mathbf{x} - \mathbf{x}_0) = 0$. Applying the Green's theorem, we have

$$\int\int_{r \geq \varepsilon} \ln r \Delta\varphi dx dy = \int\int_{r \geq \varepsilon} \Delta \ln r \varphi dx dy + \oint_{r=\varepsilon} \frac{\partial\varphi}{\partial r} \ln r ds - \oint_{r=\varepsilon} \varphi \frac{1}{r} ds. \tag{4.5}$$

By using (4.3), we have

$$\int\int_{r \geq \varepsilon} \Delta \ln r \varphi dx dy = 0. \tag{4.6}$$

As for the other terms

$$\oint_{r=\varepsilon} \frac{\partial \varphi}{\partial r} \ln r ds \leq 2\pi \left(\sup_{r=\varepsilon} \left| \frac{\partial \varphi}{\partial r} \right| \right) \varepsilon \ln \varepsilon, \tag{4.7}$$

and

$$\oint_{r=\varepsilon} \varphi \frac{1}{r} ds = \frac{1}{\varepsilon} \oint_{r=\varepsilon} \varphi ds = 2\pi \varphi(\varepsilon \cos \xi, \varepsilon \sin \xi), \tag{4.8}$$

where $\varphi(\varepsilon \cos \xi, \varepsilon \sin \xi)$ is the mean value of φ on the circle of radius ε. In the limit as $\varepsilon \to 0$, $\varphi(\varepsilon \cos \xi, \varepsilon \sin \xi) \to \varphi(0)$. Therefore,

$$\langle -\Delta \phi, \varphi \rangle \;=\; \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int \int_{r \geq \varepsilon} \ln r \Delta \varphi dx dy = \;\varphi(0) \;=\; \langle \delta(\mathbf{x} - \mathbf{x}_0), \varphi(x) \rangle. \tag{4.9}$$

Hence, we may write

$$-\Delta \phi = \delta(\mathbf{x} - \mathbf{x}_0). \tag{4.10}$$

■

Theorem 4 *The Euler equations (1.3) possess the weak singular solutions*

$$u = \frac{y - y_0}{2\pi r^2} + \dot{x}_0, \tag{4.11a}$$

$$v = -\frac{x - x_0}{2\pi r^2} + \dot{y}_0, \tag{4.11b}$$

$$p = \frac{x^2 + y^2 - 2xx_0 - 2yy_0}{8\pi^2 r^2 (x_0^2 + y_0^2)} + k\left[\frac{\ln(x_0^2 + y_0^2) - \ln r^2}{4\pi} + \dot{y}_0 x - \dot{x}_0 y \right] - \ddot{x}_0 x - \ddot{y}_0 y, \tag{4.11c}$$

where $x_0 \equiv x_0(t)$, $y_0 \equiv y_0(t)$ *are arbitrary functions of* t.

Proof. By using (4.3) and (4.11) in the distribution sense, we can get

$$\begin{aligned}
&\Omega = \Delta \phi = -\delta(\mathbf{x} - \mathbf{x}_0), \\
&u = \frac{y - y_0}{2\pi r^2} + \dot{x}_0, \\
&v = -\frac{x - x_0}{2\pi r^2} + \dot{y}_0, \\
&p_x = -u_t - uu_x - vu_y + kv = \frac{x - x_0}{4\pi^2 r^4} - k\left[\frac{x - x_0}{2\pi r^2 - \dot{y}_0} \right] - \ddot{x}_0, \\
&p_y = -v_t - uv_x - vv_y - ku = \frac{y - y_0}{4\pi^2 r^4} - k\left[\frac{y - y_0}{2\pi r^2 + \dot{x}_0} \right] - \ddot{y}_0.
\end{aligned} \tag{4.12}$$

Furthermore, we have

$$
\begin{aligned}
\Omega_x &= -\delta_x(\mathbf{x}-\mathbf{x}_0) = -\delta_r(\mathbf{x}-\mathbf{x}_0)\frac{x-x_0}{r},\\
\Omega_y &= -\delta_y(\mathbf{x}-\mathbf{x}_0) = -\delta_r(\mathbf{x}-\mathbf{x}_0)\frac{y-y_0}{r},\\
\Omega_t &= \delta_r(\mathbf{x}-\mathbf{x}_0)\frac{\dot{x}_0(x-x_0)}{r} + \delta_r(\mathbf{x}-\mathbf{x}_0)\frac{\dot{y}_0(y-y_0)}{r},
\end{aligned}
\tag{4.13}
$$

and

$$
\begin{aligned}
p &= \int_0^x p_x(x,0)dx + \int_0^y p_y(x,y)dy\\
&= \frac{x^2+y^2-2xx_0-2yy_0}{8\pi^2 r^2(x_0^2+y_0^2)} + k\left[\frac{\ln(x_0^2+y_0^2)-\ln r^2}{4\pi} + \dot{y}_0 x - \dot{x}_0 y\right] - \ddot{x}_0 x - \ddot{y}_0 y,
\end{aligned}
\tag{4.14}
$$

which integral path does not go through the point (x_0, y_0). Then, we have

$$
\begin{aligned}
\Omega_t + u\Omega_x + v\Omega_y =& \frac{\delta_r(\mathbf{x}-\mathbf{x}_0)}{r}\left[\dot{x}_0(x-x_0) + \dot{y}_0(y-y_0 - \dot{y}_0(y-y_0) - \dot{x}_0(x-x_0)\right]\\
&+ \frac{\delta_r(\mathbf{x}-\mathbf{x}_0)}{2\pi r^3}\left[(x-x_0)(y-y_0) - (x-x_0)(y-y_0)\right] = 0.
\end{aligned}
\tag{4.15}
$$

We omit the straightforward verification that the functions satisfies the 2D incompressible Euler equations with Coriolis force in the sense of distribution. ■

5 Conclusion

In this paper, we construct two families of exact solutions to the 2D incompressible Euler equations with Coriolis force. In the first family, the solutions are global and smooth with respect to the spatial variables x and y, but unbounded. In the second family, the solutions are singular, and weak in the distribution sense. A question is whether there exists bounded peaked weak solutions as in the Camassa-Holm equations. This question seems difficult but worth considering in future papers.

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